

THE QUADRIFOVAL VARIETY

LUKE OEDING

ABSTRACT. Multi-view Geometry is reviewed from an Algebraic Geometry perspective and multi-focal tensors are constructed as equivariant projections of the Grassmannian. A connection to the principal minor assignment problem is made by considering several flatlander cameras. The ideal of the quadrifocal variety is computed up to degree 8 (and partially in degree 9) using the representations of $\mathrm{GL}(3)^{\times 4}$ in the polynomial ring on the space of $3 \times 3 \times 3 \times 3$ tensors. Further representation-theoretic analysis gives a lower bound for the number of minimal generators.

1. INTRODUCTION AND BACKGROUND

1.1. The multi-view variety. Multi-view Geometry is a branch of Computer Vision [22]. An important task in Computer Vision is to efficiently reconstruct the 3-dimensional scene from the 2-dimensional projections. Typically, one first estimates the multi-focal tensor associated to the n views using correspondences arising from one object seen in multiple images. From the multi-focal tensor one reconstructs the camera matrices. After the camera matrices are known, one uses the point correspondences to triangulate the 3D points.

In the standard pinhole camera model, the projection of 3D world points to multiple 2D images is represented by a collection of 3×4 matrices (A_1, \dots, A_n) and the mapping

$$(1) \quad \begin{aligned} \mathbb{P}^3 &\rightarrow \mathbb{P}^2 \times \dots \times \mathbb{P}^2 \\ [x] &\mapsto ([A_1x], \dots, [A_nx]). \end{aligned}$$

For cameras in general position the multi-view mapping (1) defines a 3 dimensional subvariety of the Cartesian product of projective spaces called the *multi-view* variety. Aholt, Sturmfels, and Thomas demonstrated rich algebraic geometry arising from this construction in [5]. They utilized a certain Hilbert scheme to describe the multi-view variety, its defining ideal, and further algebraic properties. Their theoretical techniques included Borel fixed monomial ideals, a universal Gröbner basis, degeneration to a special monomial ideal, and more. This work catalyzed a new area coined “Algebraic Vision” by Sameer Agarwal and Rekha Thomas. The reader may wish to consult the following other examples of recent work in this field [3, 1, 2, 3, 4, 25].

1.2. Moduli spaces and quadrifocal tensors. Suppose the entries of the camera matrices are not known, but are considered as parameters. By moding out by the projective rescaling in each camera plane we have a moduli space of camera matrices. The algebraic varieties of multi-focal tensors are models for these moduli spaces, and we want to know their basic algebraic properties. Bifocal tensors are just 3×3 matrices of rank 2, defined by the 3×3 determinant. Chris Aholt and the author resolved the long-standing open question of describing the ideal of trifocal tensors [4], building on work of Alzati and Tortora [6]. They also computed its algebraic degree and Hilbert polynomial using Maple, Macaulay2 [17], and Bertini [7].

In this paper we will be mainly concerned with the *quadrifocal variety*. One may record the correspondences induced from one 3D point seen in 4 images by the 81 special 4×4 minors of stacked camera matrix $A = (A_1^\top | A_2^\top | \cdots | A_n^\top)$ that only use one column from each of the first four blocks of A . These coordinates are linear in each block, yielding a $3 \times 3 \times 3 \times 3$ *quadrifocal* tensor. The quadrifocal variety is the Zariski closure in \mathbb{P}^{80} of the set of quadrifocal tensors.

We seek a complete description of the polynomial defining equations of the quadrifocal variety. The main result of the present article is a first step in this direction.

Theorem* 1.1. *Let I_d denote the degree d piece of the ideal of the quadrifocal variety.*

I_d is zero for $d < 3$.

I_3 is 600-dimensional.

I_4 is 48,600-dimensional but contains no minimal generators.

I_5 is 1,993,977-dimensional and contains at least 1,377 minimal generators.

I_6 is 54,890,407-dimensional and contains at least 37,586 minimal generators.

I_7 is 1,140,730,128-dimensional and likely contains no minimal generators.

I_8 is 18,940,147,947-dimensional and contains at least 162,000 minimal generators.

I_9 is $\geq 223,072,284,455$ -dimensional and contains at least 3,087,000 minimal generators.

The star refers to the fact that some of our computations were done using random points of the quadrifocal variety, so the dimensions reported are only upper bounds, but the lower bounds hold *with high probability*. In Section 6 we give an invariant description of all these equations and computational evidence that the equations reported here are likely *not* a minimal set of generators.

Until now, it was only known that a quadrifocal tensor must adhere to 51 non-linear constraints [47]. Indeed, the quadrifocal variety has codimension 51, so there must be at least 51 equations, but our results show that it is very far from being a complete intersection. For instance, there are 600 cubic minimal generators. In Section 5 we give a simple description of these equations via contractions. From the contraction description we see that the cubic equations are a consequence of the fact that every contraction of a quadrifocal tensor is a homography tensor [46]. Additional equations are needed to take the set of tensors having that property alone and cut it down to the quadrifocal locus.

1.3. Multi-focal tensors in general. If π is a partition of 4, the entries of a multi-focal tensor of profile π are given by the minors of A that use π_i columns from block i . Multi-focal tensors record correspondences between multiple images. An *epipole* is the image of one camera's focal point seen in another view. Correspondences between pairs of image points (in 2 views) leads to the bi-focal tensor, or *fundamental matrix*. Whereas a point-point-line correspondences (in 3 views) are encoded by a *trifocal tensor* [15], and correspondences of quadruples of image points (in 4 views) are encoded by a *quadrifocal tensor* [47]. See [20, 21, 26, 52, 49] for applications of quadrifocal tensors.

Each multi-focal tensor can be determined by observing some minimal number of correspondences in multiple images. From 7 point correspondences in 2 images one can reconstruct the bifocal tensor (fundamental matrix). 5 point correspondences in 3 or 4 images suffice to determine the trifocal and quadrifocal tensors [22, Parts III&IV]. For a summary of these and other minimal problems in Computer Vision, see [42], and in particular one can check [27, 32, 36, 37, 43] for recent algorithms for the relative pose problems in 3 and 4 views.

Since algorithms for determining the relative pose of 2 cameras exist (for instance [34]) one may ask what is the advantage of considering more than 2 views at a time. The first

advantage is that fewer 3D points need to be identified for more than 2 views, which can be useful when many points in one view become outliers for another. In the 3 view case, the trifocal tensor can be used in the structure from motion problem when the cameras move along a straight line (see [11, 50, 31]). Other advantages include greater stability, and the possibility to avoid certain unstable or critical configurations. In the 4 view case, while the quadrifocal tensor is more difficult to construct, one may avoid iterative algorithms which are not guaranteed to converge [22, Ch. 1]. Moreover, unlike in the 3 view case where one view plays a special role, in the 4 view case all 4 images can play the same role, and it is expected that this symmetry can be exploited. For 3D to 2D projections and 4 or more views there are no n -focal tensors if $n > 4$, hence we only consider the 2, 3 and 4 view cases.

1.4. Outline. For the reader's convenience we collect notation in Section 2. We give a uniform presentation of multi-focal tensors from the viewpoint of Algebraic Geometry via equivariant projections of a Grassmannian in Section 3, which connects our work to [14, 20]. Section 4 addresses the case of different dimensional cameras and contains a connection between the multi-focal variety and the variety of principal minors of square matrices, connecting this work to [33]. We discuss contractions and homography tensors in Section 5, and state Proposition 5.3 describing the 600 bihomogeneous cubics that are the minimal generators of the quadrifocal ideal in the lowest degree. We use symmetry-enhanced calculations to determine the quadrifocal ideal up to degree 8 and partially compute the ideal in degree 9 in Section 6. In addition, we use representation theory and computations utilizing `SchurRings` to determine necessary minimal generators of the quadrifocal ideal. We conjecture that these necessary minimal generators also suffice.

2. NOTATION

Let U and V denote vector spaces, which we always consider the complex numbers, denoted \mathbb{C} , to be our ground field. The direct sum of U and V is denoted $U \oplus V$, and their tensor product is denoted $U \otimes V$. The k -th exterior power of V , or the k -mode skew-symmetric tensors is denoted $\bigwedge^k V$. Its elements are linear combinations of k -fold wedge products $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ with $v_i \in V$. We discuss alternating tensors in more depth in Section 3.2.1. The vector space of symmetric d -mode tensors is denoted $S^d V$. If we choose a basis x_1, \dots, x_m of V we may consider $S^d V$ as the vector space of homogeneous degree d polynomials on x_1, \dots, x_m . The symmetric algebra on V is denoted $Sym^\bullet V = \bigoplus_{d \geq 0} S^d V$, and is isomorphic to the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. The tensor algebra on V is denoted $V^\otimes = \bigoplus_{d \geq 0} V^{\otimes d}$.

The vector space dual, denoted V^* , is the space of all linear functionals $V \rightarrow \mathbb{C}$. After bases are chosen for U and V , $U^* \otimes V$ may be thought of as the space of matrices representing linear mappings $U \rightarrow V$. If $M \in U^* \otimes V$ is a matrix $\bigwedge^k M \in \bigwedge^k U^* \otimes \bigwedge^k V$ may be thought of as the matrix whose entries are the $k \times k$ minors of M . The general (special) linear group of all invertible (determinant 1) linear transformations of V is denoted $GL(V)$ (resp. $SL(V)$).

We use the notation $\pi \vdash d$ to denote a partition $\pi = (p_1, p_2, \dots, p_n)$ with $\sum_i p_i = d$. We let $S_\pi V$ to denote the corresponding Schur module, which we think of as an explicit representative of an irreducible submodule of the d -fold tensor product $V^{\otimes d}$. Given vector spaces V_1, V_2, V_3, V_4 and multi-partition $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ with $\pi \vdash d$ we use the shortened notations $S_\pi V$ and $S_{\pi_1} S_{\pi_2} S_{\pi_3} S_{\pi_4}$ for $S_{\pi_1} V_1 \otimes S_{\pi_2} V_2 \otimes S_{\pi_3} V_3 \otimes S_{\pi_4} V_4$.

The projective space of all lines through the origin in V is denoted $\mathbb{P}V$. If $v \in V$ is nonzero the line through v is denoted $[v]$. If $X \subset \mathbb{P}U$ and $Y \subset \mathbb{P}V$ are algebraic varieties

their Cartesian product, denoted $X \times Y$ is a subvariety of $\mathbb{P}(U \oplus V)$. The cone over the projective $X \subset \mathbb{P}V$, denoted \hat{X} is an affine subvariety of V .

Multifocal tensors sometimes need a mixture of indices and multi-indices. For instance, fundamental matrices are indexed by a pair of double indices: the notation $(F_{\{i,j\},\{k,l\}})$ indicates a matrix with rows indexed by the double index $\{i,j\}$, and columns indexed by the double index $\{k,l\}$, while $(T_{i,j,\{k,l\}})$ denotes a 3-mode tensor with the first two modes indexed by i and j respectively, with the third mode indexed by the double index $\{k,l\}$.

3. EPIPOLES, FUNDAMENTAL MATRICES, TRIFOCAL AND QUADRIFOCAI TENSORS

3.1. The multi-view setup. Multiple view geometry arises when one considers many images taken of the same scene, from (possibly) different viewpoints and is beautifully presented in [22]. The following introduction is an invariant view inspired by the ideas in [22, Ch. 17]. Let A_j denote 3×4 camera matrices (non-degenerate) for $1 \leq j \leq n$, with row spaces equal to V_j , a three-dimensional vector space.

Let W denote a 4-dimensional vector space, whose projectivization represents the 3-dimensional “projective world”. For fixed camera matrices A_j , the multi view map (which also appeared in [5]) is

$$(2) \quad \mathbb{P}W \xrightarrow{(A_1, \dots, A_n)} \mathbb{P}V_1 \times \dots \times \mathbb{P}V_n.$$

Now we wish to treat the camera matrices as variable or as having indeterminate entries. The map in (2) is the same if we replace the matrices A_j with scalar multiples of themselves. So, we should consider our space of parameters for cameras to be

$$\mathbb{P}(W^* \otimes V_1) \times \dots \times \mathbb{P}(W^* \otimes V_n) \quad n\text{-camera space.}$$

We note here that if different camera models are taken, this paradigm may be easily altered to accommodate such changes by altering the spaces in which the cameras are modeled.

3.2. The Grassmannian. Faugeras and Mourrain studied multi-view geometry from the point of view of the Grassmann algebra, see [14, 20]. We also adapt that approach as it provides a uniform treatment and a convenient way to organize many of our computations.

3.2.1. Exterior products. Recall if V is a vector space with basis u_1, \dots, u_n , we construct the exterior powers of U , denoted $\bigwedge^k U$ by considering the alternating (or wedge) product \wedge and forming the vector space of k -vectors (length k wedge products) with basis consisting of pure k -vectors $\{u_{i_1}, \dots, u_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$. Thus $\bigwedge^k U$ has dimension $\binom{n}{k}$ if $0 \leq k \leq n$ and 0 otherwise. It is straightforward to see that $v_1 \wedge \dots \wedge v_k \neq 0$ if and only if the vectors $\{v_1, \dots, v_k\}$ are linearly independent in U . Consider two non-zero k -vectors $v_1 \wedge \dots \wedge v_k$ and $w_1 \wedge \dots \wedge w_k$ and the underlying vector spaces $E := \text{span}\{v_1, \dots, v_k\}$ and $F := \text{span}\{w_1, \dots, w_k\}$. It is straightforward to check that

$$v_1 \wedge \dots \wedge v_k = \lambda(w_1 \wedge \dots \wedge w_k) \quad \text{for some } \lambda \neq 0 \quad \text{if and only if} \quad \bigwedge^k E = \bigwedge^k F,$$

and equality holds when λ is the determinant of the change of basis between E and F .

We denote by $\mathbb{P}\bigwedge^k U$ the projective space consisting of lines through the origin in $\bigwedge^k U$, which we may consider as the set of classes $[\omega] = \{\lambda\omega \mid \lambda \in \mathbb{C} \setminus \{0\}, \omega \in \bigwedge^k U \setminus \{0\}\}$.

This motivates the definition of the Grassmannian (in its minimal embedding). Let $\text{Gr}(k, U)$ denote the set of k -planes in U . The rational map:

$$\begin{aligned} \text{Gr}(k, U) &\rightarrow \mathbb{P}\bigwedge^k U \\ M &\mapsto \bigwedge^k M \end{aligned}$$

is an embedding (in fact, the embedding is a minimal rational embedding). The usual Plücker embedding is a slight variant of this construction, which we will review next in the context of multiple view geometry.

3.2.2. From multiple views to the Grassmannian. It is natural to consider the following $4 \times 3n$ blocked matrix, which will present a convenient way to keep track of external constraints on the multi-view setup

$$M = (A_1^\top \mid A_2^\top \mid \dots \mid A_n^\top) \in (W^* \otimes V_1) \oplus \dots \oplus (W^* \otimes V_n) = W^* \otimes (\bigoplus_j V_j).$$

The non-degeneracy condition is that each matrix A_j in an n -tuple $([A_1^\top], [A_2^\top], \dots, [A_n^\top])$ must have full rank, which occurs in an open set; so the row space of M parameterizes the (Grassmannian variety) 4-dimensional subspaces of a $3n$ -dimensional space.

The maximal minors of M give coordinates (the Plücker coordinates) on the Grassmannian $\text{Gr}(4, 3n)$. These minors are also known as *multilinear coordinates*.

In invariant language this parameterization is the following map

$$(3) \quad \begin{aligned} \varphi: W^* \otimes (\bigoplus_j V_j) &\longrightarrow \bigwedge^4 W^* \otimes \bigwedge^4 (\bigoplus_j V_j) \cong \bigwedge^4 (\bigoplus_j V_j) \\ M &\longmapsto \bigwedge^4 M \end{aligned}$$

The image of φ (the Zariski closure of the image of an open set) is isomorphic to the cone over the Grassmannian of 4-dimensional subspaces of $\bigoplus_j V_j$, which is the row space of M . In other words

$$\text{Im}(\varphi) = \widehat{\text{Gr}}(4, \bigoplus_j V_j) \subset \bigwedge^4 (\bigoplus_j V_j).$$

Notice that because W is 4-dimensional, $\bigwedge^4 W$ is one-dimensional, and thus passing to the maximal minors of the concatenated matrix M removes the dependency on the world points represented by $\mathbb{P}W$.

It is well known that the dimension of the Grassmannian $\text{Gr}(r, \mathbb{C}^N)$ is $r(N - r)$. In our example $\dim(\text{Gr}(4, 3n)) = 4(3n - 4)$.

If we restrict to camera space, we have to consider the image of the map up to the n -dimensional torus action which records the projective ambiguity in each of the n cameras. We can restrict the target of the map to the appropriate *GIT quotient*:

$$\mathbb{P}(W^* \otimes V_1) \times \dots \times \mathbb{P}(W^* \otimes V_n) \rightarrow \widehat{\text{Gr}}(4, \bigoplus_j V_j) // (\mathbb{C}^*)^n \subset \bigwedge^4 (\bigoplus_j V_j) // (\mathbb{C}^*)^n.$$

From this we obtain the dimension of the GIT quotient (see [22, § 17.5] and [5, § 6])

$$\dim \widehat{\text{Gr}}(4, \bigoplus_j V_j) // (\mathbb{C}^*)^n = 4(3n - 4) + 1 - n = 11n - 15.$$

Remark 3.1. Here is a classical formula for the degree of the Grassmann manifold, (see [35], [16, Ex. 14.7.11] or [13, § 10.1.2, eq. (10.6)])

$$\deg \text{Gr}(r, n) = \frac{1!2! \dots r! \dim(\text{Gr}(r, n))!}{(n - r)!(n - r + 1)! \dots n!}.$$

For example

$$\deg(\mathrm{Gr}(4, 6)) = 14, \quad \deg(\mathrm{Gr}(4, 9)) = 1662804, \quad \deg(\mathrm{Gr}(4, 12)) = 1489877926680.$$

One hope is that a better understanding of the GIT quotient of the Grassmannian might allow us to find the degree of the multi-focal tensor varieties.

3.3. Symmetry. For $j \in \{1, 2, 3, 4\}$ let A_j be a 3×4 (non-degenerate) camera matrix (an element of $W^* \otimes V_j$), with blocking $A_j = (B_j | \mathbf{x}_j)$. On each matrix A_j we have an action of $\mathrm{GL}(V_j) \cong \mathrm{GL}(3)$ acting by change of coordinates in the camera plane. The action is

$$\begin{aligned} & (\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3) \times \mathrm{GL}(V_4)) \times ((W^* \otimes V_1) \oplus (W^* \otimes V_2) \oplus (W^* \otimes V_3) \oplus (W^* \otimes V_4)) \\ & \rightarrow ((W^* \otimes V_1) \oplus (W^* \otimes V_2) \oplus (W^* \otimes V_3) \oplus (W^* \otimes V_4)) \\ & (g_1, g_2, g_3, g_4), (A_1, A_2, A_3, A_4) \mapsto (g_1 A_1, g_2 A_2, g_3 A_3, g_4 A_4), \end{aligned}$$

where we take the action of each g_j to be a change of basis in the row space of A_j . Because the matrices A_j are assumed to be full rank, we can, without loss of generality, act by an element of $\mathrm{GL}(3)^{\times 4}$ and assume that $B_j = \mathrm{Id}_3$ and move the 4-tuple $(A_1^\top | A_2^\top | A_3^\top | A_4^\top)$ to

$$(4) \quad A \cong \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} & x_{3,1} & x_{3,2} & x_{3,3} & x_{4,1} & x_{4,2} & x_{4,3} \end{array} \right).$$

Remark 3.2. There is an action of $\mathrm{GL}(W)$ acting on simultaneously on all of the column spaces of A_j (which are all equal to W). While this action doesn't turn out to be useful (it is trivialized in the tensorial map), the action of \mathfrak{S}_4 permuting the matrices A_j , also permutes the indices in the image of the tensorial map, and preserves the set of quadrifocal tensors.

3.4. Multilinearity spaces. The 4×4 minors of M come in several classes, which have invariant descriptions. By construction, $\Lambda^4(\bigoplus_j V_j)$ is a vector space with a natural $\mathrm{GL}(3n)$ -action, but we can further view this as a $\mathrm{GL}(\Lambda^4(\bigoplus_j V_j))$ -action (each $\mathrm{GL}(V_j)$ acting by invertible linear change of coordinates in V_j), and there is a natural inclusion of $G := \prod_j \mathrm{GL}(V_j)$, which may be thought of as block diagonal (with the proper choice of basis) inside $\mathrm{GL}(3n)$. Moreover, we may view G as a product of a torus $\mathbb{T}^n := \prod_j T_j \cong (\mathbb{C}^*)^n$ ($T_j \cong \mathbb{C}^*$ acting by scaling block j of M) and $\prod_j \mathrm{SL}(V_j)$. In summary

$$G := \prod_j T_j \times \prod_j \mathrm{SL}(V_j) \subset \mathrm{GL}(\Lambda^4 \bigoplus V_j).$$

Thus, we may consider $\mathrm{Gr}(4, \bigoplus_j V_j)$ as a G -variety. On the other hand, on the GIT quotient, the torus action is trivialized, so we will consider $G' := \prod_j \mathrm{SL}(V_j)$ acting on $\mathrm{Gr}(4, \bigoplus_j V_j) \subset \Lambda^4 \bigoplus V_j$ and on the GIT quotient $\widehat{\mathrm{Gr}}(4, \bigoplus_j V_j) // (\mathbb{C}^*)^n \subset \Lambda^4(\bigoplus_j V_j) // (\mathbb{C}^*)^n$. The effect of trivializing the torus action is that we may identify every V_j with its dual, and this induces an identification of every irreducible representation $S_\pi V_j$ with its dual $S_\pi V_j$.

Now G acts on $\Lambda^4(\bigoplus_j V_j)$ and it has a decomposition into irreducible G -modules:

$$\begin{aligned} \Lambda^4 \left(\bigoplus_j V_j \right) = & \left(\bigoplus_{i \neq j} \Lambda^3 V_i \otimes V_j \right) \oplus \left(\bigoplus_{i < j} \Lambda^2 V_i \otimes \Lambda^2 V_j \right) \\ & \oplus \left(\bigoplus_{i \notin \{j,k\}, j < k} \Lambda^2 V_i \otimes V_j \otimes V_k \right) \oplus \left(\bigoplus_{i < j < k < l} V_i \otimes V_j \otimes V_k \otimes V_l \right). \end{aligned}$$

The 4 non-isomorphic module classes and their descriptions are listed in Table 1.

Now we will consider the projection of the cone over the Grassmannian $\widehat{\mathrm{Gr}}(4, \bigoplus_j V_j)$ to each type of multi-linearity space. The images of the projections are respectively the single

Columns used	Invariant description	Space
$i_1 \ i_2 \ i_3 \ j$	$\bigwedge^3 V_i \otimes V_j \cong V_j$	<i>epipole space</i>
$i_1 \ i_2 \ j_1 \ j_2$	$\bigwedge^2 V_i \otimes \bigwedge^2 V_j \cong V_i^* \otimes V_j^*$	<i>fundamental matrix space</i>
$i_1 \ i_2 \ j \ k$	$\bigwedge^2 V_i \otimes V_j \otimes V_k \cong V_i^* \otimes V_j \otimes V_k$	<i>trifocal space</i>
$i \ j \ k \ l$	$V_i \otimes V_j \otimes V_k \otimes V_l$	<i>quadrifocal space</i>

TABLE 1. *Classes of multi-focal tensors.* Assume i, j, k, l are distinct, i_s are in block i , and j_s are in block j .

$S = \begin{pmatrix} (x_{1,1}-x_{2,1}) & (-1)(x_{1,2}-x_{2,2}) & (x_{1,3}-x_{2,3}) \end{pmatrix}^\top$	$(1, 2)$ -epipole
$F = \begin{pmatrix} 0 & x_{1,3}-x_{2,3} & -x_{1,2}+x_{2,2} \\ -x_{1,3}+x_{2,3} & 0 & x_{1,1}-x_{2,1} \\ x_{1,2}-x_{2,2} & -x_{1,1}+x_{2,1} & 0 \end{pmatrix}$	$(1, 2)$ -fundamental matrix
$\left. \begin{aligned} T_{i,i,\{k,l\}} &= -x_{1,i}+x_{2,i} \\ T_{i,k,\{i,l\}} &= -x_{1,i}+x_{3,i} \\ T_{k,i,\{i,l\}} &= x_{2,i}-x_{3,i} \\ T_{i,j,\{k,l\}} &= 0 \end{aligned} \right\} \begin{aligned} &k, l \text{ distinct,} \\ &i \text{ distinct from } k \& l, \\ &\text{else.} \end{aligned}$	$(1, 2, 3^*)$ -trifocal tensor
$\left. \begin{aligned} Q_{i,i,k,l} &= (-1)^i(x_{1,i}-x_{2,i}) = -Q_{i,i,l,k}, \\ Q_{i,k,i,l} &= (-1)^i(x_{1,i}-x_{3,i}) = -Q_{i,l,i,k}, \\ Q_{i,k,l,i} &= (-1)^i(x_{1,i}-x_{4,i}) = -Q_{i,l,k,i}, \\ Q_{k,i,i,l} &= (-1)^i(x_{2,i}-x_{3,i}) = -Q_{l,i,i,k}, \\ Q_{k,i,l,i} &= (-1)^i(x_{2,i}-x_{4,i}) = -Q_{l,i,k,i}, \\ Q_{k,l,i,i} &= (-1)^i(x_{3,i}-x_{4,i}) = -Q_{l,k,i,i} \\ Q_{i,j,k,l} &= 0 \end{aligned} \right\} \begin{aligned} &k < l, \\ &i \text{ distinct from } k \& l, \\ &\text{else.} \end{aligned}$	$(1, 2, 3, 4)$ -quadrifocal tensor

TABLE 2. *Parametrizations of multi-focal tensors up to symmetry.*

view, the epipolar variety, the trifocal variety, and the quadrifocal variety. The fiber of the projection over a general point is the product of the ignored camera planes and the torus acting on the utilized camera planes. So it is interesting to consider the minimal number of cameras in each case. Moreover, because the projections on the level of vector spaces are equivariant, the images are automatically invariant (with respect to the appropriate group).

Suppose now that there are at most 4 cameras. For each $i \in \{1, 2, 3, 4\}$ let the set $\{e_1^i, e_2^i, e_3^i\}$ denote a basis of V_i , which also provides an ordered basis on $\mathbb{C}^{12} \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4$. In the remainder of this section we consider the parameterization of epipoles Sec. 3.5, fundamental matrices Sec. 3.6, trifocal tensors Sec. 3.7, and quadrifocal tensors Sec. 3.8, all up to the G -action using the matrix A in (4). These results are summarized in Table 2.

3.5. Epipoles. For a pair of cameras, we consider the projection of $\text{Gr}(4, 6) = \text{Gr}(4, V_1 \oplus V_2) \subset \mathbb{P}\bigwedge^4(V_1 \oplus V_2) = \mathbb{P}^{14}$ to epipolar space $\mathbb{P}(V_1 \otimes \bigwedge^3 V_2) = \mathbb{P}^2$. The target space is (naturally isomorphic to) the projective plane, and the map subjects onto \mathbb{P}^2 . The image is naturally $\text{GL}(V_1) \times \text{GL}(V_2)$ -invariant, the action of $\text{GL}(V_1)$ being trivial and the 2-dimensional torus acts by a weight of $(3, 1)$.

The image of the projection is the space of epipoles in view 1 imposed by view 2. Thus the epipoles may be recovered from the multi-view setup via a projection from the Grassmannian.

To get an expression of an epipole in coordinates consider just two cameras and the matrix

$$A \cong \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} \end{array} \right).$$

An element S of single camera space has the following form in the Plücker coordinates associated to the determinants of the matrices constructed from one column from the first block of A the three columns of the second block:

$$S = \sum_{1 \leq p \leq 3} S_{p,\{1,2,3\}} e_p^1 \wedge e_1^2 \wedge e_2^2 \wedge e_3^2 \in \bigwedge^3 V_2 \otimes V_1 \cong V_1.$$

Applying this to A we get the coordinates of the epipole

$$\begin{pmatrix} S_{1,\{1,2,3\}}(A) \\ S_{2,\{1,2,3\}}(A) \\ S_{3,\{1,2,3\}}(A) \end{pmatrix} = \begin{pmatrix} (x_{1,1} - x_{2,1}) \\ (-1)(x_{1,2} - x_{2,2}) \\ (x_{1,3} - x_{2,3}) \end{pmatrix}.$$

3.6. Fundamental matrices. Again for a pair of cameras, we may consider the projection of $\text{Gr}(4, 6) = \text{Gr}(4, V_1 \oplus V_2) \subset \mathbb{P} \bigwedge^4 (V_1 \oplus V_2) = \mathbb{P}^{14}$ to fundamental matrix space $\mathbb{P}(V_1^* \otimes V_2^*) = \mathbb{P}^8$. The target space may be interpreted as the projectivization of a space of 3×3 matrices, and carries the natural action of $\text{GL}(V_1) \times \text{GL}(V_2)$. We might call the image of the projection the *variety of fundamental matrices*, which is also naturally $\text{GL}(V_1) \times \text{GL}(V_2)$ -invariant. Because the vector spaces V_1 and V_2 play symmetric roles, this image is also naturally \mathfrak{S}_2 invariant. It is well known that the matrices in the image of the projection have a one-dimensional kernel and the image variety is just the (degree 3) determinantal hypersurface.

Note that $\widehat{\text{Gr}}(4, 6)$ is 9-dimensional, the GIT quotient $\widehat{\text{Gr}}(4, 6) // (\mathbb{C}^*)^2$ is 7-dimensional, and thus birational to the projective variety of 3×3 matrices of rank ≤ 2 , the variety of fundamental matrices. Also note that the 2-dimensional torus acts by a weight of $(2, 2)$.

An element F of fundamental matrix space has the following form in Plücker coordinates:

$$F = \sum_{1 \leq i, j, k, l \leq 3, i < j, k < l} F_{\{i, j\}, \{k, l\}} e_i^1 \wedge e_j^1 \wedge e_k^2 \wedge e_l^2 \in \bigwedge^2 V_i \otimes \bigwedge^2 V_j \cong V_i^* \otimes V_j^*.$$

For a fixed pair of distinct indices i, j , a fundamental matrix is gotten by applying F to a blocked 4×12 camera matrix A . In particular, $F(A)$ may be thought of as a vector whose coordinates are determinants of the 4×4 submatrices of A obtained by taking two columns from each from blocks i and j of A .

Now apply F to A in (4). The (1-2) fundamental matrix associated to A is described by coordinates $F_{\{i, j\}, \{k, l\}}$, and we can represent this as the matrix

$$F(A) = \begin{pmatrix} 0 & x_{1,3} - x_{2,3} & -x_{1,2} + x_{2,2} \\ -x_{1,3} + x_{2,3} & 0 & x_{1,1} - x_{2,1} \\ x_{1,2} - x_{2,2} & -x_{1,1} + x_{2,1} & 0 \end{pmatrix},$$

where the (p, q) entry is $F_{\{i, j\}, \{k, l\}}$ such that $\{p, i, j\} = \{q, k, l\} = \{1, 2, 3\}$.

3.7. The trifocal variety. For a triple of cameras, we consider the projection of $\text{Gr}(4, 9) = \text{Gr}(4, V_1 \oplus V_2 \oplus V_3) \subset \mathbb{P} \bigwedge^4 (V_1 \oplus V_2 \oplus V_3) = \mathbb{P}^{(9)} - 1$ to trifocal space $\mathbb{P}(V_1 \otimes V_2 \otimes V_3^*) = \mathbb{P}^{26}$. The target space may be interpreted as the projectivization of a space of $3 \times 3 \times 3$ tensors, and carries the natural action of $\text{GL}(V_1) \times \text{GL}(V_2) \times \text{GL}(V_3)$. The image of the projection is called the *trifocal variety*, which is also naturally $\text{GL}(V_1) \times \text{GL}(V_2) \times \text{GL}(V_3)$ -invariant. Because the vector spaces V_1 and V_2 play symmetric roles, this image is also naturally \mathfrak{S}_2 invariant.

Non-trivial permutations involving V_3 will not preserve this trifocal variety but produce an isomorphic copy it.

The trifocal variety is 18-dimensional ([22, p. 368]). The GIT quotient $\widehat{\text{Gr}(4, 9)} // (\mathbb{C}^*)^3$ is 18-dimensional and thus is birational to the trifocal variety. In this case, the 3-dimensional torus acts by a weight of $(1, 1, 2)$.

For the trifocal tensor, consider 3 cameras in special position producing

$$A \cong \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} & x_{3,1} & x_{3,2} & x_{3,3} \end{array} \right).$$

An element T of trifocal space has the following form in Plücker coordinates:

$$T = \sum_{1 \leq i, j, k, l \leq 3, k < l} T_{i,j,\{k,l\}} e_i^1 \wedge e_j^2 \wedge e_k^3 \wedge e_l^3 \in V_1 \otimes V_2 \otimes \wedge^2 V_3 \cong V_1 \otimes V_2 \otimes V_3^*.$$

A trifocal tensor is gotten by applying T to A . In particular, $T(A)$ may be thought of as a vector whose coordinates are determinants of the 4×4 submatrices of A obtained by taking one column from each of the first two blocks of A and two columns of from the third block of A .

Lemma 3.3. *The locus of trifocal tensors is the $\text{GL}(3) \times \text{GL}(3) \times \text{GL}(3)$ -orbit of the 9-dimensional linear space whose coordinates are given by $T_{i,j,\{k,l\}}$ satisfying the following conditions:*

$$\left. \begin{array}{l} T_{i,i,\{k,l\}}(A) = -x_{1,i} + x_{2,i} \\ T_{i,k,\{i,l\}}(A) = -x_{1,i} + x_{3,i} \\ T_{k,i,\{i,l\}}(A) = x_{2,i} - x_{3,i} \\ T_{i,j,\{k,l\}}(A) = 0 \end{array} \right\} \begin{array}{l} k \text{ and } l \text{ distinct, and } i \text{ distinct from } k, l, \\ \text{else.} \end{array}$$

3.8. The quadrifocal variety. For a quadruple of cameras, we consider the projection of $\text{Gr}(4, 12) = \text{Gr}(4, V_1 \oplus V_2 \oplus V_3 \oplus V_4) \subset \mathbb{P} \wedge^4 (V_1 \oplus V_2 \oplus V_3 \oplus V_4) = \mathbb{P}^{\binom{12}{4}-1}$ to quadrifocal space $\mathbb{P}(V_1 \otimes V_2 \otimes V_3 \otimes V_4) = \mathbb{P}^{80}$. The target space may be interpreted as the projectivization of a space of $3 \times 3 \times 3 \times 3$ tensors, and carries the natural action of $\text{GL}(V_1) \times \text{GL}(V_2) \times \text{GL}(V_3) \times \text{GL}(V_4)$. The image of the projection is called the *quadrifocal variety*, which is also naturally $\text{GL}(V_1) \times \text{GL}(V_2) \times \text{GL}(V_3) \times \text{GL}(V_4)$ -invariant. Because the vector spaces V_1, V_2, V_3 and V_4 play symmetric roles, this image is also naturally \mathfrak{S}_4 invariant.

The quadrifocal variety is 29-dimensional ([22, p. 423]), and the GIT quotient $\widehat{\text{Gr}(4, 12)} // (\mathbb{C}^*)^4$ is 29-dimensional, and thus birational to the quadrifocal variety. In this case, the 4-dimensional torus acts by a weight of $(1, 1, 1, 1)$.

An element Q of quadrifocal space has the following form in Plücker coordinates:

$$Q = \sum_{1 \leq i, j, k, l \leq 3} Q_{i,j,k,l} e_i^1 \wedge e_j^2 \wedge e_k^3 \wedge e_l^4 \in V_1 \otimes V_2 \otimes V_3 \otimes V_4.$$

A quadrifocal tensor is gotten by applying Q to a blocked 4×12 camera matrix A . In particular, $Q(A)$ may be thought of as a vector whose coordinates are determinants of the 4×4 submatrices of A obtained by taking one column from each of the 4 blocks of A .

For the quadrifocal tensor, consider 4 cameras in special position.

$$A \cong \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} & x_{3,1} & x_{3,2} & x_{3,3} & x_{4,1} & x_{4,2} & x_{4,3} \end{array} \right).$$

Lemma 3.4. *The locus of quadrifocal tensors is the $\mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(3)$ -orbit of the 12-dimensional linear space of tensors whose coordinates are given by $Q_{i,j,k,l}$ satisfying the following conditions:*

$$\left. \begin{aligned} Q_{i,i,k,l}(A) &= (-1)^i(x_{1,i} - x_{2,i}) = -Q_{i,i,l,k}(A), \\ Q_{i,k,i,l}(A) &= (-1)^i(x_{1,i} - x_{3,i}) = -Q_{i,l,i,k}(A), \\ Q_{i,k,l,i}(A) &= (-1)^i(x_{1,i} - x_{4,i}) = -Q_{i,l,k,i}(A), \\ Q_{k,i,i,l}(A) &= (-1)^i(x_{2,i} - x_{3,i}) = -Q_{l,i,i,k}(A), \\ Q_{k,i,l,i}(A) &= (-1)^i(x_{2,i} - x_{4,i}) = -Q_{l,i,k,i}(A), \\ Q_{k,l,i,i}(A) &= (-1)^i(x_{3,i} - x_{4,i}) = -Q_{l,k,i,i}(A) \end{aligned} \right\} \quad \begin{array}{l} k < l \text{ and } i \text{ distinct from } k, l, \\ \text{else.} \end{array}$$

$$Q_{i,j,k,l}(A) = 0$$

4. ARBITRARY DIMENSIONAL CAMERAS: A CONNECTION TO PRINCIPAL MINORS

It is curious to study the case when W and V_i arbitrary dimensions (see [9]). For instance, when each V_i has dimension two, we might imagine the cameras to be flatlanders' cameras. When the V_i have dimension more than 3, we might consider an image plane that records higher dimensional data such as temperature or color.

4.1. Principal minors. In the case $V_i \cong \mathbb{C}^2$ we find it very interesting that seemingly unrelated work regarding principal minors fits into this framework [23, 33, 10, 39, 38, 18, 19].

Suppose W has dimension m and that we have m copies of V_i , each with dimension 2. Consider again the matrix $M = (A_1^\top \mid A_2^\top \mid \dots \mid A_m^\top)$, now as a $m \times 2m$ matrix consisting of m blocks of size $m \times 2$. By a left action of $\mathrm{GL}(W)$ we may assume that each block of M is of the form $A_i^\top = (e_i \mid b_i)$, where e_i is the i -th standard basis vector of W and b_i is arbitrary. Let B denote the $m \times m$ matrix with columns b_i .

It is straightforward to see that the maximal minors of M correspond to the minors of B and moreover that the minors of M that use precisely one column from each block of M correspond to the principal minors of B .

In turn, this identification naturally gives the space $V_1 \otimes \dots \otimes V_m$ the interpretation as the space of all 2^m principal minors of an $m \times m$ matrix (the 0×0 minor may be assumed to be 1 in this construction).

The case $m = 4$ was studied by [10, 33], who discovered the minimal defining equations of the ideal of relations among principal minors of a generic 4×4 matrix, and found that the algebraic variety coincides with the main component of the singular locus of the $2 \times 2 \times 2 \times 2$ hyperdeterminant.

Remark 4.1. Weyman and Zelevinski [51] analyzed the singular locus of the hyperdeterminant. In the $2 \times 2 \times 2 \times 2$ case, there are 8 components of the singular locus:

$$\nabla_{\mathrm{cusp}} \cup \nabla_{\mathrm{node}}(\emptyset) \cup \nabla_{\mathrm{node}}(\{i, j\}) \quad 1 \leq i < j \leq 4.$$

Moreover, according to Holweck, [24] the “main” component $\nabla_{\mathrm{node}}(\emptyset)$ also has the interpretation as the projective dual of the secant line variety:

$$\nabla_{\mathrm{node}}(\emptyset) = \sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)^\vee \subset \mathbb{P}(V_1 \otimes V_2 \otimes V_3 \otimes V_4)^*.$$

The variety of principal minors of 4×4 matrices is the projection

$$\mathrm{Gr}(4, V_1 \oplus V_2 \oplus V_3 \oplus V_4) \rightarrow \mathbb{P}(V_1 \otimes V_2 \otimes V_3 \otimes V_4).$$

The dual variety of $\mathrm{Gr}(4, 8)$ is *not* one of the 3 cases which is normal [24], however it does have many other nice properties, such as a finitely generated invariant ring, finitely many

orbits in the null cone, and a set of normal forms that depend on 8 parameters, [12]. It would be nice to see how to exploit these coincidences.

In the case that the matrix B is assumed to be symmetric, [23] re-invigorated a classical study going back to Cayley of relations amongst principal minors of symmetric matrices. (See [33] and [39] for a historical discussion.) In particular, the ideal of relations of the principal minors of a generic symmetric 3×3 matrix is generated by a single equation known as Cayley's $2 \times 2 \times 2$ hyperdeterminant, denoted h for this discussion. In the 4×4 case Holtz and Sturmfels discovered that the $\mathfrak{S}_4 \times \prod_{i=1..4} \mathrm{SL}(V_i)$ -orbit of h generates the ideal of relations, and conjectured that a similar orbit of equations, which they called the hyperdeterminantal module, generates the ideal of relations among principal minors of a symmetric matrix. This conjecture was solved (set-theoretically) by the author [39, 38]. We note that in the framework of this paper, imposing that the matrix B be symmetric naturally imposes a restriction from the Grassmannian $\mathrm{Gr}(m, 2m)$ to the Lagrangian Grassmannian $\mathrm{LGr}(m, 2m)$, (see [38]).

In the case that the matrix B is assumed to be skew-symmetric, one considers the principal Pfaffians. It turns out that the relations among principal Pfaffians are precisely the equations of the orthogonal Grassmannian, which are analogous to the Plücker relations and were known classically, (see [29] for a modern treatment in invariant language).

4.2. Higher dimensional images. It would be interesting to consider, for instance, a camera with fixed position continuously viewing a scene in time in our paradigm as W a 5-dimensional vector space and V_i each as 4-dimensional vector spaces (3 space dimensions and one time). This seems to be a promising approach to understanding this and a wide variety of generalizations.

5. CONTRACTIONS AND HOMOGRAPHY TENSORS

According to Shashua and Wolf [47] a contraction of a quadrifocal tensor is a homography tensor [46] in the other 3 views. Moreover, this property defines the set of homography tensors. Similarly, the contraction of a homography tensor is an LLC mapping (a rank 2 matrix relating the epipoles in two views). We will give an invariant description of this process and show how to get 600 independent internal constraints on quadrifocal tensors.

Consider $V_1 \otimes V_2 \otimes V_3 \otimes V_4$ and suppose we choose a basis of V_4 , $\{x, y, z\}$. Then we may write $Q \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ as

$$Q(x, y, z) = xQ_1 + yQ_2 + zQ_3,$$

where Q_i are the standard $3 \times 3 \times 3$ slices of the $3 \times 3 \times 3 \times 3$ tensor. By abuse of notation, we also think of $Q(x, y, z)$ as a function from \mathbb{C}^3 to $V_1 \otimes V_2 \otimes V_3$, and let x, y, z act as variables.

The following results come from [47] and [46].

Theorem 5.1 ([47, Theorem 2]). *If Q is a quadrifocal tensor then $Q(x, y, z)$ is a homography tensor for every value of x, y, z .*

For our purposes, the above result can be used to define the notion of “homography tensor.” Now choose a basis a, b, c of V_3 . If $H \in V_1 \otimes V_2 \otimes V_3$, then we may write $H(a, b, c) = aH_1 + bH_2 + cH_3$. In slightly different language [4, Prop. 7.2] we showed that the Zariski closure of the homography tensors is an irreducible variety (we called it $\mathrm{P-Rank}^{2,2,2}$), defined (at least set-theoretically) by 30 cubic equations.

Theorem 5.2 ([47, Theorem 1]). *If H is a homography tensor then $H(a, b, c)$ is an LLC (Linear Line Complex) mapping for every value of a, b, c .*

For our purposes, an LLC is a skew-symmetric 3×3 matrix, which necessarily has (even) rank ≤ 2 . Thus for all values of a, b, c the matrix $H(a, b, c)$ must satisfy the constraint $\det(H(a, b, c)) \equiv 0$. In particular, every coefficient in the cubic polynomial in a, b, c must vanish. This condition gives a basis of the ten-dimensional space of cubics. Note, the coefficient of a^3 is the determinant of the first slice, and the coefficients on b^3 and c^3 are respectively the determinants of the second and third slices. This space also has the interpretation of $\bigwedge^3 V_1^* \otimes \bigwedge^3 V_2^* \otimes S^3 V_3^*$ as a G -module.

We can apply the same method to quadrifocal tensors. Namely if Q is a quadrifocal tensor, $Q(x, y, z)$ must satisfy all of the internal trifocal constraints for all values of x, y, z . In particular, $Q(x, y, z)(a, b, c)$ must be an epipolar matrix (whose entries are bi-homogeneous quadrics), and we must have the bi-homogeneous sextic polynomial (of bi-degree (3,3)) $\det(Q(x, y, z)(a, b, c)) \equiv 0$ for all values of x, y, z, a, b, c .

The space of bi-degree (3,3) sextics is 100-dimensional, and the coefficients of the expression $\det(Q(x, y, z)(a, b, c))$ provide a basis of a 100-dimensional space of cubics that vanish on the quadrifocal variety. Note, the coefficient of $x^3 a^3$ is the determinant of the first slice, and there are 8 other monomials that are the product of two cubes, the coefficients of which correspond to the determinants of the 8 other slices. This space also has the interpretation as the $\mathrm{GL}(3)^{\times 4}$ -module $\bigwedge^3 V_1 \otimes \bigwedge^3 V_2 \otimes S^3 V_3 \otimes S^3 V_4$.

If we interchange the roles of V_1, V_2, V_3, V_4 , and apply the same construction we obtain 6 non-isomorphic modules of the same format $\bigwedge^3 V_i \otimes \bigwedge^3 V_j \otimes S^3 V_k \otimes S^3 V_l$. In particular, we find a space of 600 cubic polynomials in the ideal of the quadrifocal variety, 54 of which are determinants of 3×3 slices of a $3 \times 3 \times 3 \times 3$ tensor.

Let $G := \mathfrak{S}_4 \ltimes \mathrm{GL}(3)^{\times 4}$. Since the quadrifocal variety is G -invariant, we can describe its ideal as a G module. For convenience, when the \mathfrak{S}_4 symmetry is present, we write $S_{\pi_1} S_{\pi_2} S_{\pi_3} S_{\pi_4}$ for the direct sum of $S_{\pi_1} V_1^* \otimes S_{\pi_2} V_2^* \otimes S_{\pi_3} V_3^* \otimes S_{\pi_4} V_4^*$ and all non-redundant copies of it obtained by permuting the indices.

The above discussion implies the following:

Proposition 5.3. *Suppose Q is a quadrifocal tensor. Then the 600 polynomials forming a basis of the G -module $S_3 S_3 S_{1,1,1} S_{1,1,1}$ vanish on Q .*

Because one of the contractions of a trifocal tensor also form a homography, we know that 10 cubic polynomials vanish on the set of trifocal tensors. In [4] we showed that this condition cuts out a subset of the $3 \times 3 \times 3$ tensors consisting of 4 irreducible algebraic varieties. In order to distinguish the trifocal variety, more equations are needed.

Theorem 5.4 ([4]). *The ideal of the trifocal variety in $V_1 \otimes V_2 \otimes V_3^*$ is generated by 10 cubic, 81 quintic, and 1980 sextic polynomials. These are represented by the following $\mathrm{GL}(3)^{\times 3}$ modules:*

$$\mathcal{M}_3 = \bigwedge^3 V_1^* \otimes \bigwedge^3 V_2^* \otimes S^3 V_3,$$

$$\mathcal{M}_5 = S_{221} V_1^* \otimes S_{221} V_2^* \otimes S_{311} V_3 \oplus S_{221} V_1^* \otimes S_{221} V_2^* \otimes S_{221} V_3,$$

$$\begin{aligned} \mathcal{M}_6 = & S_{411} V_1^* \otimes S_{33} V_2^* \otimes S_{222} V_3 \oplus S_{33} V_1^* \otimes S_{411} V_2^* \otimes S_{222} V_3 \oplus S_{33} V_1^* \otimes S_{222} V_2^* \otimes S_{411} V_3 \\ & \oplus S_{222} V_1^* \otimes S_{33} V_2^* \otimes S_{411} V_3 \oplus S_{33} V_1^* \otimes S_{33} V_2^* \otimes S_{222} V_3 \oplus S_{33} V_1^* \otimes S_{222} V_2^* \otimes S_{33} V_3 \\ & \oplus S_{222} V_1^* \otimes S_{33} V_2^* \otimes S_{33} V_3 \oplus S_{33} V_1^* \otimes S_{321} V_2^* \otimes S_{321} V_3 \oplus S_{321} V_1^* \otimes S_{33} V_2^* \otimes S_{321} V_3. \end{aligned}$$

graded piece	$\dim I_d$	necessary G modules of minimal generators	dimension of necessary mingens
I_2	0	$\mathcal{M}_2 = 0$	0
I_3	600	$\mathcal{M}_3 = S_3 S_3 S_{1,1,1} S_{1,1,1}$	600
I_4	48,600	$\mathcal{M}_4 = 0$	0
I_5	1,993,977	$\mathcal{M}_5 = S_{3,1,1} S_{3,1,1} S_{3,1,1} S_{3,1,1}$ $\oplus S_{2,2,1} S_{2,2,1} S_{2,2,1} S_{2,2,1}$	1,377
I_6	54,890,407	$\mathcal{M}_6 = S_{4,1,1} S_{3,3} S_{2,2,2} S_{2,2,2} \otimes \mathbb{C}^2$ $\oplus S_{3,3} S_{3,3} S_{2,2,2} S_{2,2,2} \otimes \mathbb{C}^2$ $\oplus S_{3,2,1} S_{3,2,1} S_{2,2,2} S_{2,2,2}$ $\oplus S_{2,2,2} S_{2,2,2} S_{2,2,2} S_{2,2,2} \otimes \mathbb{C}^2$ $\oplus S_6 S_{3,3} S_{3,3} S_{2,2,2}$	37,586
I_7	1,140,730,128	$\mathcal{M}_7 = 0$	0
I_8	18,940,147,947	$\mathcal{M}_8 = S_{4,4} S_{4,4} S_{4,4} S_{4,2,2} \otimes \mathbb{C}^2$	162,000
I_9	$\geq 223,072,284,455$	$\mathcal{M}_9 \geq S_{5,4} S_{5,4} S_{5,4} S_{4,3,2}$ $\oplus S_{5,4} S_{5,4} S_{5,4} S_{5,2,2}$	$\geq 3,087,000$

TABLE 3. The dimension and isotypic description of the ideal of the quadrifocal variety up to degree 9.

In the next section we work to obtain a similar statement for quadrifocal tensors.

6. COMPUTATIONAL RESULTS FOR THE QUADRIFOVAL IDEAL

The goal of this section is to give a description of the lowest degree part of the vanishing ideal for the quadrifocal variety in terms of G -modules. First recall that the polynomial ring $\bigoplus_d S^d(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$ has a graded isotopic decomposition (see [30], for instance) with respect to $G = \mathfrak{S}_4 \ltimes \mathrm{GL}(3)^{\times 4}$:

$$S^d(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*) = \bigoplus_{\pi} S_{\pi} V^* \otimes \mathbb{C}^{m_{\pi}},$$

with Schur modules $S_{\pi} V^* := S_{\pi_1} V_1^* \otimes S_{\pi_2} V_2^* \otimes S_{\pi_3} V_3^* \otimes S_{\pi_4} V_4^*$ and multiplicity space $\mathbb{C}^{m_{\pi}}$. The multiplicity space $\mathbb{C}^{m_{\pi}}$ has a basis given by linear combinations of fillings of shape π . We write $(S_{\pi_1} S_{\pi_2} S_{\pi_3} S_{\pi_4}) \otimes \mathbb{C}^m$ to indicate the G module (occurring with multiplicity m) gotten by summing over all non-redundant permutations of the partitions indexing $S_{\pi} V^*$.

Computation 6.1. *Let I_d denote the degree d piece of the ideal of the quadrifocal variety, and let $G = \mathfrak{S}_4 \ltimes \mathrm{GL}(3)^{\times 4}$. The symmetry assisted computations of I_d up to degree $d \leq 9$ using random points of the quadrifocal variety are reported in Table 3. For $5 \leq d \leq 9$ the dimensions of I_d and the list necessary modules of minimal generators in Table 3 are correct with high probability.*

Description of computation: The qualifier “with high probability” refers to the fact that we computed the ideal on random subsets of points from the quadrifocal variety, so in principle we could have chosen a set of points in special position and would have over counted the dimension of the ideal. The set of points in such special position, however, being of lower dimension, has measure zero in the quadrifocal variety, so we say the results hold with high probability. The proof techniques we use for the degree ≤ 4 computations, however are valid unconditionally.

The content of this computation is two-fold. First we computed (in Maple) the entire ideal degree by degree, up to degree 8 and partially in degree 9, making use of the isotypic decomposition of the polynomial ring. Second, we checked for representation-theoretic certificates for necessity of minimal generators using **SchurRings** in Macaulay2. In the ancillary files associated with the arXiv version of this manuscript we provide a minimal set of data necessary to check our work. This includes basic maple scripts, fillings that yield a basis of each isotypic component, and the resulting modules in the ideal. Here is a summary of these computations.

Every irreducible G -module has the property that it is the vector space spanned by the G -orbit of a so-called *highest weight vector* (see [28]). We used Young symmetrizers to obtain a basis of each multiplicity space \mathbb{C}^{m_π} in the isotypic decomposition of the polynomial ring above. Then we computed the subspace of \mathbb{C}^{m_π} that vanished on the quadrifocal tensors. This algorithm was used and outlined in [4, 8] for example. Since we learned this algorithm from the paper of Landsberg and Manivel [30], we call this algorithm the *Landsberg-Manivel-algorithm* or the *LM-algorithm* for short.

The output of this symmetry-enhanced polynomial interpolation computation is a G -module description of the ideal in each degree. The dimensions of these modules give the beginning of the Hilbert function of the quadrifocal ideal, and are also reported in Table 3. We will use notation of **SchurRings**: The ring $Sym^\bullet(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$ is regarded as a tower of rings. The variables in each ring are represented as s_π , (respectively t_π, u_π, v_π), for partitions π . The correspondence between the two notations is

$$ms_{\pi_1}t_{\pi_2}u_{\pi_3}v_{\pi_4} \leftrightarrow S_{\pi_1}V_1^* \otimes S_{\pi_2}V_2^* \otimes S_{\pi_3}V_3^* \otimes S_{\pi_4}V_4^* \otimes \mathbb{C}^m.$$

In degree 3 our application of the LM-algorithm found the following modules:

$$I_3 = (s_{(1,1,1)}t_{(1,1,1)}u_3 + (s_{(1,1,1)}t_3 + s_3t_{(1,1,1)})u_{(1,1,1)})v_3 + ((s_{(1,1,1)}t_3 + s_3t_{(1,1,1)})u_3 + s_3t_3u_{(1,1,1)})v_{(1,1,1)}.$$

To save space, we only record those modules up to the \mathfrak{S}_4 -action:

$$\mathcal{M}_3 := s_3t_3u_{(1,1,1)}v_{(1,1,1)}$$

This module corresponds to the same 600 polynomials in Proposition 5.3. Non-vanishing of polynomials on random points of a variety implies non-vanishing for the entire variety, so this 600 dimensional vector space of cubics are the only cubics vanishing on the quadrifocal variety. Therefore, the $d \leq 3$ computations hold with no “high probability” qualifier.

The modules we found (by the LM-algorithm) in degree 4 are (up to the \mathfrak{S}_4 -action):

$$I_4 = (s_4t_4 + (s_4 + s_{(3,1)})t_{(3,1)})u_{(2,1,1)}v_{(2,1,1)}$$

Because $81 * 600 = 48600$, we guess that all the equations in I_4 come from linear combinations of products of linear forms with the 600 cubics in I_3 and thus we would guess that there are no new generators in degree 4. We would like to prove this using representation theory.

The multiplication in the ring $S^\bullet(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$ is just the usual polynomial multiplication. It is not easy to determine the isotopic version of multiplication. However, we can get a lower bound on the modules of minimal generators in our ideal. The basic idea is the following. The multiplication in the ring $S^\bullet(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$ is also the restriction of the multiplication in the tensor ring $(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)^{\otimes}$. The tensor product of two representations of the form $S_\pi V^*$ is obtained by an iteration of the Littlewood-Richardson rule.

Using the package `SchurRings` [48] we computed the tensor product

$$I_3 \otimes (V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$$

and found that it exactly coincides with I_4 . This is an indication that there are probably no new generators in degree 4, however it could be that the modules resulting in $I_3 \otimes V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*$ are not actually in $S^4(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$. The only way that there could be new minimal generators in degree 4 is if there were already some syzygies amongst the degree 3 equations. At least in degree 4 it is possible to look at the highest weight vectors and check that they are in the ideal generated by the 600 cubics in Macaulay2. On the other hand, we can argue in a less computationally intensive way by using the following special case of a more general idea from [45], which was employed in [44, 40].

Lemma 6.2. *Suppose $F_\pi := (F_{\pi_1}, F_{\pi_2}, F_{\pi_3}, F_{\pi_4})$ is a filling using the ordered alphabets $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ giving a nonzero realization of the module $S_\pi V^*$ in $S^d(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$. If $F_\mu := (F_{\mu_1}, F_{\mu_2}, F_{\mu_3}, F_{\mu_4})$ is a filling giving a nonzero realization of the module $S_\mu V^*$ in $S^{d-1}(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$ and F_μ may be obtained from F_π by respectively deleting the last used letter in each of the alphabets $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$, then $S_\pi V^*$ is in the ideal generated by $S_\mu V^*$.*

Now we apply Lemma 6.2 to see that $S_{211}S_{211}S_{31}S_{31}$ is in the ideal generated by $S_{111}S_{111}S_3S_3$. In this case everything occurs with multiplicity one, so our work is much easier. The filling

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}$$

produces a realization of a copy of $S_{31}S_{31}S_{211}S_{211}$ in the ideal of the quadrifocal variety. Notice that by removing the fourth letter we obtain the filling

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array},$$

which produces a nonzero copy of $S_{111}S_{111}S_3S_3$.

The same argument may be applied to $S_4S_4S_{211}S_{211}$, with the same result. Therefore, none of the modules in degree 4 are minimal generators of the ideal of the quadrifocal variety.

In degree 5 we note that $1929501 - 600 * 3321 = 1,377$. This means that the degree 3 equations cannot generate all of the ideal in degree 5, and there must be at least 1,377 new minimal generators in degree 5. Moreover, if there are no degree 2 syzygies amongst the degree 3 equations then the space of minimal generators in degree 5 would be precisely 1,377-dimensional. In principle one could try to use a degree-limited Gröbner basis computation in Macaulay2 to check if each highest weight vector of each of the modules in I_5 is actually in $\langle I_3 \rangle_5$, but memory limitations become a problem.

However, we can argue by comparing multiplicities of isotypic components. In degree 5 the LM-algorithm produced the following modules (up to the \mathfrak{S}_4 -action):

$$\begin{aligned} I_5 = & (s_5 t_5 + (s_5 + 2s_{(4,1)})t_{(4,1)} + (s_5 + s_{(4,1)} + s_{(3,2)})t_{(3,2)} + (3s_{(4,1)} + 7s_{(3,1,1)})t_{(3,1,1)})u_{(3,1,1)}v_{(3,1,1)} \\ & + (((s_5 + 2s_{(4,1)})t_{(4,1)} + s_{(4,1)}t_{(3,2)} + (2s_5 + 2s_{(4,1)} + 2s_{(3,2)})t_{(3,1,1)})u_{(3,1,1)} \\ & + (s_5 t_5 + (s_5 + 2s_{(4,1)})t_{(4,1)} + (s_5 + s_{(4,1)} + s_{(3,2)})t_{(3,2)} + (s_{(4,1)} + s_{(3,1,1)})t_{(3,1,1)} + s_{(2,2,1)}t_{(2,2,1)})u_{(2,2,1)}v_{(2,2,1)}. \end{aligned}$$

Using **SchurRings** we found that the only modules left in the difference between I_5 and $I_4 \otimes V_1 \otimes V_2 \otimes V_3 \otimes V_4$ are

$$\mathcal{M}_5 := s_{(3,1,1)}t_{(3,1,1)}u_{(3,1,1)}v_{(3,1,1)} + s_{(2,2,1)}t_{(2,2,1)}u_{(2,2,1)}v_{(2,2,1)}.$$

From this computation we know that these two summands must be among the minimal generators. We do not know if there are any other minimal generators in degree 5.

Remark 6.3. While this **SchurRings** computation ignores the structure of the multiplicity spaces in the ideals, to first approximation it tells if there is any representation-theoretic reason for modules to be among the minimal generators.

In degree 6 the LM-algorithm produced the following modules (up to the \mathfrak{S}_4 -action) $I_6 =$

$$\begin{aligned} & (s_6t_6 + (s_6 + 2s_{(5,1)})t_{(5,1)} + (s_6 + 2s_{(5,1)} + 2s_{(4,2)})t_{(4,2)} + (3s_{(5,1)} + 3s_{(4,2)} + 10s_{(4,1,1)})t_{(4,1,1)})u_{(4,1,1)}v_{(4,1,1)} \\ & + ((s_6 + s_{(5,1)} + s_{(4,2)})t_{(4,1,1)}u_{(4,1,1)} + s_{(4,1,1)}t_{(4,1,1)}u_{(3,3)})v_{(3,3)} \\ & + (((s_6 + 3s_{(5,1)})t_{(5,1)} + (s_6 + 3s_{(5,1)} + 3s_{(4,2)})t_{(4,2)} + (2s_6 + 7s_{(5,1)} + 7s_{(4,2)} + 12s_{(4,1,1)})t_{(4,1,1)})u_{(4,1,1)} \\ & + (s_{(5,1)} + s_{(4,2)} + 2s_{(4,1,1)})t_{(4,1,1)}u_{(3,3)} \\ & + (s_6t_6 + (2s_6 + 6s_{(5,1)})t_{(5,1)} + (2s_6 + 6s_{(5,1)} + 6s_{(4,2)})t_{(4,2)} + (3s_6 + 10s_{(5,1)} + 10s_{(4,2)} + 16s_{(4,1,1)})t_{(4,1,1)} \\ & + (s_6 + 2s_{(5,1)} + 2s_{(4,2)} + 4s_{(4,1,1)} + s_{(3,3)})t_{(3,3)} \\ & + (3s_6 + 12s_{(5,1)} + 12s_{(4,2)} + 16s_{(4,1,1)} + 3s_{(3,3)} + 24s_{(3,2,1)})t_{(3,2,1)})u_{(3,2,1)})v_{(3,2,1)} \\ & + ((s_{(5,1)}t_{(5,1)} + (s_{(5,1)} + s_{(4,2)})t_{(4,2)} + (s_6 + 3s_{(5,1)} + 3s_{(4,2)} + 6s_{(4,1,1)})t_{(4,1,1)})u_{(4,1,1)} \\ & + (s_{(4,1,1)}t_{(4,1,1)} + s_6t_{(3,3)})u_{(3,3)} + ((s_6 + 3s_{(5,1)})t_{(5,1)} + (s_6 + 3s_{(5,1)} + 3s_{(4,2)})t_{(4,2)} \\ & + (s_6 + 4s_{(5,1)} + 4s_{(4,2)} + 5s_{(4,1,1)})t_{(4,1,1)} + (s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)})t_{(3,3)} \\ & + (s_6 + 4s_{(5,1)} + 4s_{(4,2)} + 6s_{(4,1,1)} + 3s_{(3,3)} + 8s_{(3,2,1)})t_{(3,2,1)})u_{(3,2,1)} \\ & + (s_6t_6 + 2s_{(5,1)}t_{(5,1)} + (s_6 + s_{(5,1)} + 2s_{(4,2)})t_{(4,2)} \\ & + (2s_{(5,1)} + 2s_{(4,2)} + 2s_{(4,1,1)})t_{(4,1,1)} + (s_{(5,1)} + 3s_{(4,1,1)} + 3s_{(3,3)})t_{(3,3)} \\ & + (s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + 4s_{(3,2,1)})t_{(3,2,1)} + 3s_{(2,2,2)}t_{(2,2,2)})u_{(2,2,2)})v_{(2,2,2)}. \end{aligned}$$

Applying the same **SchurRings** test we find the following necessary minimal generators:

$$\mathcal{M}_6 := (s_6t_{(3,3)}u_{(3,3)} + ((2s_{(4,1,1)} + 2s_{(3,3)})t_{(3,3)} + s_{(3,2,1)}t_{(3,2,1)} + 2s_{(2,2,2)}t_{(2,2,2)})u_{(2,2,2)})v_{(2,2,2)}.$$

These modules could not come from any of the lower degree parts of the ideal, so they are among the minimal generators in degree 6, however there could be more minimal generators that we haven't accounted for if there were syzygies among the lower degree generators.

The space of degree 7 polynomials $S^7(V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*)$ decomposes as a sum of 288 isotypic components (up to \mathfrak{S}_4 symmetry) with dimension of multiplicity spaces as large as 301. Thus applying the LM-algorithm directly in degree 7 was more challenging and we were forced to exploit a parallelism, following a process similar to that in [41]. We ran the LM-algorithm in a separate instance of Maple for each of the 288 isotypic components. This allowed us to distribute the computation. Running in parallel the degree 7 computation took approximately 12 hours on our two servers with respectively 40 and 24 processors. We found 201 modules occurring non-trivially the ideal. The **SchurRings** test yielded no new necessary minimal generators.

The parallelized LM-algorithm for degree 8 involved 619 Schur modules with multiplicity spaces up to 608-dimensional. Parts of this computation took approximately 20 days on our

servers. We found 453 modules occurring in I_8 with nonzero multiplicity. The **ShurRings** test only produced one module occurring in I_8 with greater multiplicity than can occur in $I_7 \otimes V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*$, namely

$$\mathcal{M}_8 := 2s_{(4,4)}t_{(4,4)}u_{(4,4)}v_{(4,2,2)}.$$

That this module is not in the ideal generated by $\mathcal{M}_3 + \mathcal{M}_5 + \mathcal{M}_6$ can be seen for shape reasons alone. The Schur module \mathcal{M}_8 is indexed by a multi-partition containing three partitions of shape $(4, 4)$, but no module the list of known minimal generators is indexed by quadruple of partitions with a subset of 3 of them that each fit in a box of dimensions 2×4 . Therefore \mathcal{M}_8 must occur with multiplicity 2 in the minimal generators in the ideal.

In degree 9 there are 1205 isotypic components, with multiplicity spaces as high as 2226-dimensional. After approximately one month of computational time on our servers we were able to evaluate 1158 of these isotypic components on the quadrifocal variety, 951 of which occurred non-trivially in the ideal. The computations that finished were those modules for which the multiplicity was low (under around 300).

Remark 6.4. The main obstruction to completing the degree 9 computation is a lack of available machines able to perform Maple computations. More specifically, if a given isotypic component has multiplicity m in the polynomial ring, we must populate an $m \times m$ matrix with m^2 evaluations (from the Young symmetrizer algorithm), once to verify we have found a basis of the space, and a second time to evaluate this basis on points of the quadrifocal variety. Each of these evaluations could be done in a separate instance. So in the case of multiplicity 2226, we must perform approximately 10 million evaluations, which could potentially be done on a separate processor if were to take fuller advantage of the parallel nature of this problem. One of the largest multiplicity spaces in degree 9 we were able to handle was 456 dimensional, and the computation in multiplicity 2226 case is approximately 23 times larger than this. Since this scale of resources is not currently available to us, it seems unlikely that we will be able to complete the degree 9 computation.

Though we were only able to partially compute I_9 , we were able to obtain two new modules in degree 9 that must be among the minimal generators of the quadrifocal ideal:

$$\widetilde{\mathcal{M}}_9 := s_{(5,4)}t_{(5,4)}u_{(5,4)}v_{(4,3,2)} + s_{(5,4)}t_{(5,4)}u_{(5,4)}v_{(5,2,2)}.$$

These two modules occur with multiplicity 3 in I_9 , but instances of these modules coming from $I_8 \otimes V_1^* \otimes V_2^* \otimes V_3^* \otimes V_4^*$ occurred with multiplicity 2. Therefore, these modules must occur with multiplicity at least 1 among the minimal generators of I .

The dimension counts in Table 3 are a straightforward application of a hook length formula, which is also implemented in **SchurRings** for example.

We wondered if Table 3 might be a complete list of minimal generators for the quadrifocal ideal. We performed further experiments with modules in degrees 9, 10, and 11 having low multiplicity. We found that $S_{5,5}S_{5,5}S_{5,5}S_{4,3,3}$ occurred with multiplicity one greater in I_{10} than what could come from the part of I_9 that we were able to compute. This was the only new module we found in low multiplicity in degree 10. Further experiments suggest that there may be many more new modules of minimal generators with low multiplicity in degree 11. At present we are not confident enough to form a conjecture as to what happens in higher degree, nor in which degree the last set of minimal generators must occur.

Remark 6.5. In a previous draft we had incorrectly computed the number of equations in the ideal because of an unfortunate programming error.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, AUBURN, AL

E-mail address: oeding@auburn.edu